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A SUPERREFLEXIVE BANACH SPACE X WITH $L(X)$ ADMITTING A HOMOMORPHISM ONTO THE BANACH ALGEBRA $C(\beta N)$

BY
P. MANKIEWICZ
Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland

ABSTRACT
A separable superreflexive Banach space X is constructed such that the Banach algebra $L(X)$ of all continuous endomorphisms of X admits a continuous homomorphism onto the Banach algebra $C(\beta N)$ of all scalar valued functions on the Stone–Čech compactification of the positive integers with supremum norm. In particular: (i) the cardinality of the set of all linear multiplicative functionals on $L(X)$ is equal to 2^c and (ii) X is not isomorphic to any finite Cartesian power of any Banach space.

1. Introduction

The first results showing the existence of a nontrivial linear multiplicative functional acting on the Banach algebra $L(X)$ of all continuous linear endomorphisms of some space X were obtained by B. S. Mityagin and I. C. Edelstein in [10]. Namely, they proved the existence of such a functional acting on the Banach algebra $L(J)$, where J is the well known space constructed by R. C. James in [5] and on the Banach algebra $L(C(\Gamma_{\omega_1}))$, where $C(\Gamma_{\omega_1})$ is the space of all continuous scalar valued functions on the set of ordinals not exceeding the first uncountable ordinal with its usual order topology, equipped with the supremum norm (cf. [12]). A generalization of this result was given by A. Wilansky in [18]. Recently, another construction of a Banach space X with $L(X)$ admitting a nontrivial linear multiplicative functional was given by S. Shelah and J. Steprans [13]. No examples of that kind were known with the underlying Banach space being reflexive. A very simple and well known

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argument yields that the existence of a nontrivial linear multiplicative functional on the Banach algebra of all continuous endomorphisms of a Banach space X implies that X is not isomorphic to any finite Cartesian power of any Banach space (cf. Remark 6.4). Some examples of superreflexive Banach spaces not isomorphic to their Cartesian squares have been known, [2], while an example of a real superreflexive Banach space not isomorphic to the Cartesian square of any Banach space was constructed in [15]. The problem of constructing a complex variant of the example above and the existence of a superreflexive Banach space not isomorphic to any finite Cartesian power of any Banach space were still open (cf. [15], Problems 7.4 and 7.6).

It seems that the main obstacle in constructing a nontrivial linear multiplicative functional on a Banach algebra $L(X)$ lies in the fact that $L(X)$ is "strongly noncommutative". Let us mention, by the way, that no examples were known with $L(X)$ admitting more than one such functional. The problem discussed here motivates the following natural generalizations:

Problem A. Does there exist a Banach space X with $L(X)$ admitting a Banach algebra continuous homomorphism onto a "relatively large" commutative Banach algebra $\mathscr{B}$.

Note that the results of [10] and [18] imply the existence of such a homomorphism for some Banach spaces onto a one-dimensional Banach algebra.

Problem B. The same as in Problem A but with X being reflexive or even superreflexive.

The aim of this note is to prove the following

Theorem 1.1. *There exists a separable superreflexive Banach space Y with the properties:*

(i) *Y has a finite dimensional decomposition,*

(ii) *$L(Y)$ admits a continuous homomorphism onto the Banach algebra $C(\beta\mathbf{N})$,*

(iii) *for every $t \in \mathbf{R}$ there is a projection $P_t \in L(Y)$ and a linear multiplicative functional φ_t on $L(Y)$ such that for every $t_1, t_2 \in \mathbf{R}$*

$$\varphi_{t_1}(P_{t_2}) = \begin{cases} 1 & \text{for } t_1 = t_2, \\ 0 & \text{otherwise,} \end{cases}$$

where $C(\beta\mathbf{N})$ denotes the Banach algebra of all continuous scalar valued *functions on the Stone–Čech compactification of positive integers equipped with supremum norm.*

Note that Theorem 1.1 solves Problem B with $\mathscr{B} = C(\beta\mathbf{N})$, and thus yields the solutions to Problems 7.4 and 7.6 from [15], while the properties (i) and (iii) ensure that $L(Y)$ is not "too small".

The construction of the space satisfying Theorem 1.1 is done in two steps. First we prove the existence of finite dimesional Banach spaces with some strange properties (Proposition 3.2). This is a generalization of a result from [9] (Proposition 2.1 below). We would like to stress that the underlying argument here is based on the technique of "random finite dimensional Banach spaces" introduced by E. D. Gluskin in [3] and developed in various contexts by several authors ([1], [4], [7]–[9], [14]–[16]). In the second step we apply the procedure of "glueing together" such spaces presented in [15] and [16] and finally we use an ultrafilter argument to get Theorem 1.1.

The paper is organised as follows: §2 explains notations and presents known results. §3 contains the proof of the basic finite dimensional result. In §4 the construction of a space satisfying Theorem 1.1 is given. §5 is devoted to the proof of Theorem 1.1 and the last section contains concluding remarks and corollaries.

2. Preliminaries and known results

Our notation and terminology is standard. To fix the notation we shall consider real Banach spaces only. However, exactly the same argument yields all the results of this paper in the complex case except Corollary 6.3. We shall consider $\mathbf{R}^n$ equipped with different norms. If $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ then $\| x \|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \in [1, \infty)$ and $l_n^p = (\mathbf{R}^n, \| \ \|_p)$. If X is a Banach space, then by $L(X)$ we shall denote the algebra of all continuous operators acting on X and $\| T \|_X$ will stand for the norm of $T \in L(X)$. In particular we shall write $\| T \|_p$ for the norm of $T \in L(\mathbf{R}^n)$ considered as an operator acting on l_n^p. If X and Y are Banach spaces and T is a continuous linear operator from X into Y, we shall denote its norm by $\| T : X \to Y \|$. Finally, $\| T \|_{\mathrm{hs}}$ for $T \in L(\mathbf{R}^n)$ will stand for the Hilbert-Schmidt norm of T. Recall that if $X = (\mathbf{R}^n, \| \ \|_X)$ and $T \in L(\mathbf{R}^n)$ then

$$\gamma_2(T, X) = \inf\{ \| S_1 : X \to l_n^2 \| \ \| S_2 : l_n^2 \to X \| \},$$

where the infimum is taken over all pairs of operators S_1, S_2 such that $T = S_2 S_1$, and for a pair X, Y of k-dimensional Banach spaces, $k \in \mathbf{N}$, the Banach–Mazur distance

$$d(X, Y) = \inf\{ \| T : X \to Y \| \quad \| T^{-1} : Y \to X \| \},$$

where the infimum in taken over all 1–1 linear operators T acting from X onto Y.

For a linear subspace $E \subset \mathbf{R}^n$ by P_E and $\mathrm{vol}_{\dim E}$ we shall mean the orthogonal projection onto E and the usual dim E — dimensional euclidian volume on E, and B_n^p and B_X will stand for the closed unit balls in l_n^p and X respectively. The following known "local" result is crucial for the proof of Theorem 1.1 ([9], Proposition 2.3).

PROPOSITION 2.1. *There is a numerical constant $c > 0$ such that for every $n \geqq 2$ there is a norm $\| \quad \|_{X_n}$ on $\mathbf{R}^n$ such that:*

(i) *the Banach space $X_n = (\mathbf{R}^n, \| \quad \|_{X_n})$ is isometrically isomorphic to a quotient of l_{2n}^1,*

(ii) $\| x \|_2 \leqq \| x \|_{X_n} \leqq \| x \|_1 \; (\leqq \sqrt{n} \, \| x \|_2)$ *for every $x \in \mathbf{R}^n$,*

(iii) *for every $T \in L(\mathbf{R}^n)$ there is a $\lambda_T \in \mathbf{R}$ and $V_T \in L(\mathbf{R}^n)$ and a linear subspace E_T with* dim $E_T \geqq 7n/8$ *such that*

(a) $T = \lambda_T \mathrm{Id}_{\mathbf{R}^n} + V_T$,

(b) $|\lambda_T| \leqq c \, \| T \|_{X_n}$,

(c) $\| V_T \mid E_T \|_2 \leqq c \, \| T \|_{X_n} n^{-1/2}$.

If $\mathcal{B}_1$ and $\mathcal{B}_2$ are Banach algebras, by a homomorphism from $\mathcal{B}_1$ into $\mathcal{B}_2$ we shall mean a continuous linear and multiplicative map from $\mathcal{B}_1$ into $\mathcal{B}_2$. In particular, if $\mathcal{B}_2$ is the field of scalars such a homomorphism will be called a linear multiplicative functional. By $l_{\mathbf{N}}^\infty$ we shall denote the Banach algebra of all bounded scalar valued sequences with coordinatewise multiplication and its standard supremum norm. It is well known that $l_{\mathbf{N}}^\infty$ is isometrically equivalent to the Banach algebra $C(\beta\mathbf{N})$ of all scalar valued continuous functions on the Stone–Čech compactification of $\mathbf{N}$ with supremum norm.

Recall that an ultrafilter $\mathcal{U}$ on $\mathbf{N}$ is said to be a free ultrafilter iff it is not generated by a single point in $\mathbf{N}$. We shall need the following well known fact

FACT 2.2. There exists a family $\{N_t : t \in \mathbf{R}\}$ of infinite "almost disjoint" subsets of $\mathbf{N}$ (i.e. for every $t_1, t_2 \in \mathbf{R}$, $t_1 \neq t_2$ the intersection $N_{t_1} \cap N_{t_2}$ is finite).

3. Local results

We begin with the following consequence of Proposition 2.1 which seems to be of independent interest.

PROPOSITION 3.1. *There is a numerial constant $C \geqq 1$ such that for every $n \geqq 2$ and every $p \in [1, 2)$ there is a norm $\| \ \|_{X_n^p}$ on $\mathbf{R}^n$ satisfying the conditions:*

(i) *the Banach space $X_n^p = (\mathbf{R}^n, \| \ \|_{X_n^p})$ is isometrically isomorphic to a quotient of l_{2n}^p,*

(ii) $(2n)^{1/p-1} \| x \|_2 \leqq \| x \|_{X_n^p} \leqq \| x \|_p \leqq n^{1/p-1/2} \| x \|_2$ *for every $x \in \mathbf{R}^n$,*

(iii) *there is a subspace $E \subset \mathbf{R}^n$, $\dim E \geqq n/2$ such that*

$$C^{-1} n^{1/p-1/2} \| x \|_2 \leqq \| x \|_{X_n^p} \leqq n^{1/p-1/2} \| x \|_2$$

for every $x \in E$ and for every $p \in [1, 2)$,

(iv) *for every $T \in L(\mathbf{R}^n)$ there is a $\lambda_T \in \mathbf{R}$, $V_T \in L(\mathbf{R}^n)$ and a linear subspace $E_T \subset \mathbf{R}^n$, $\dim E_T \geqq 7n/8$, such that for every $p \in [1, 2)$*

(a) $T = \lambda_T \mathrm{Id}_{\mathbf{R}^n} + V_T$,

(b) $|\lambda_T| \leqq C \| T \|_{X_n^p}$,

(c) $\| V_T \mid E_T \|_2 \leqq C n^{1/2-1/p} \| T \|_{X_n^p}$.

(v) *for every $T \in L(\mathbf{R}^n)$ there is a linear subspace $F_T \subset \mathbf{R}^n$, $\dim F_T \geqq 7n/8$, such that for every $p \in [1, 2)$*

$$\| T \mid F_T \|_2 \leqq C n^{1/2-1/p} \gamma_2(T, X_n^p).$$

PROOF. We begin with the case $p = 1$. Let $\| \ \|_{X_n^1}$, for $n \geqq 2$, be the norms satisfying Proposition 2.1. Then the conditions (i), (ii) and (iv) are fulfilled with the constant c.

To prove (iii) let $Q_n : l_{2n}^1 \to X_n^1$ be the quotient map yielding the isometry from (i). Thus $B_{X_n^1} = Q_n(B_{2n}^1)$. Hence $B_{X_n^1} =$ absolute convex hull $\{Q_n e_j : j = 1, 2, \ldots, 2n\}$, where $\{e_j\}_{j=1}^{2n}$ is the standard unit vector basis in l_{2n}^1. By (ii) we have

$$\| Q_n e_j \|_2 \leqq 1 \qquad \text{for } j = 1, 2, \ldots, 2n \tag{3.1}$$

and therefore

$$[\mathrm{vol}_n(B_{X_n^1}) / \mathrm{vol}_n(n^{-1/2} B_2^n)]^{1/n} \leqq (8e^3/\pi)^{1/2}$$

(see e.g. [15], Remark 3.2). On the other hand, by (ii), we have

$$B_{X_n^1} \supset B_n^1 \supset n^{-1/2} B_n^2.$$

Now, (iii) follows by a standard "volume ratio" argument with some constant $C_1 > 1$ (cf. [17]).

To prove (v) fix an arbitrary operator $T \in L(\mathbf{R}^n)$ and let $\alpha_T = \min \| T \mid F \|_2$, where the minimum is taken over all linear subspaces $F \subset \mathbf{R}^n$ with $\dim F \geqq 7n/8$. Assume that $\alpha_T = 1$. This means that there is a linear subspace $\tilde{F} \subset \mathbf{R}^n$ with $\dim \tilde{F} \geqq n/8$ such that

$$(3.2) \qquad \| Tx \|_2 \geqq \| x \|_2 \qquad \text{for every } x \in \tilde{F}.$$

Let $S_1 : \mathbf{R}^n \to l_n^2$ and $S_2 : l_n^2 \to \mathbf{R}^n$ be such that $T = S_2 S_1$ and

$$\gamma_2(T, X_n^1) = \| S_1 : X_n^1 \to l_n^2 \| \quad \| S_2 : l_n^2 \to X_n^1 \| .$$

Replacing S_1 and S_2 by λS_1 and $\lambda^{-1} S_1$ for an appropriate $\lambda \in \mathbf{R}$ we may assume that there are linear subspaces $E_1, E_2 \subset \tilde{F}$ with $\dim E_1$, $\dim E_2 \geqq n/16$ such that

$$(3.3) \qquad \| S_1 x \|_2 \geqq \| x \|_2 \qquad \text{for every } x \in E_1$$

and

$$(3.4) \qquad \| S_1 x \|_2 \leqq \| x \|_2 \qquad \text{for every } x \in E_2.$$

Since, by (ii), $\| e_i \|_{X_n^1} = 1$ for $i = 1, 2, \ldots, n$ we infer that

$$(3.5) \qquad \| S_1 : X_n^1 \to l_n^2 \| \geqq \| S_1 : l_n^1 \to l_n^2 \| .$$

Observe that, by (3.3), we have $\| S_1 \|_{\mathrm{hs}}^2 \geqq n/16$ and hence

$$(3.6) \qquad \sup\{ \| S_1 e_i \|_2 : i = 1, 2, \ldots, n \} = \| S_1 : l_n^1 \to l_n^2 \| \geqq 1/4.$$

Combining (3.5) and (3.6) we get

$$(3.7) \qquad \| S_1 : X_n^1 \to l_n^2 \| \geqq 1/4.$$

On the other hand, by (3.2) and (3.4), $\| S_2 y \|_2 \geqq \| y \|_2$ for every $y \in S_1(E_2)$. Thus

$$\begin{aligned} \| S_2 : l_n^2 \to X_n^1 \| &\geqq \| S_2 \mid S_1(E_2) : (S_1(E_2), \| \ \|_2) \to X_n^1 \| \\ (3.8) \qquad &\geqq \| T \mid E_2 : (E_2, \| \ \|_2) \to X_n^1 \| \\ &\geqq \| T \mid E_2 : (E_2, \| \ \|_2) \to (TE_2, \| \ \|_{P_{TE_2} B_{X_n^1}}) \| . \end{aligned}$$

Now, since $P_{TE_2} B_{X_n^1}$ is equal to the absolute convex hull of vectors $P_{TE_2} Q_n e_j$, $j = 1, 2, \ldots, 2n$, each of them, by (3.1), of norm not greater than 1, a well known argument yields that

$$\mathrm{vol}_{\dim E_2} P_{TE_2} B_{X_n^1} \leqq \mathrm{vol}_{\dim E_2} c' B^1_{\dim E_2}$$

where c' is a numerical constant (cf. e.g. [7]). Obviously by (3.2),

$$\mathrm{vol}_{\dim E_2} TB_{(E_2, \| \ \|_2)} \geqq \mathrm{vol}_{\dim E_2} B^2_{\dim E_2}.$$

Both volume estimates easily imply that

$$\| T \mid E_2 : (E_2, \| \ \|_2) \to (TE_2, \| \ \|_{P_{TE_2}B_{X_n^1}}) \| \geqq c'' n^{1/2}, \tag{3.9}$$

where c'' is a numerical constant. Combining (3.8) and (3.9) we get

$$\| S_2 : l_n^2 \to X_n^1 \| \geqq c'' n^{1/2},$$

which, together with (3.7), yields that for some numerical constant C_2 we have

$$\gamma_2(T, X_n^1) \geqq C_2^{-1} n^{1/2}.$$

To complete the proof of (v), it is enough to observe that, by a standard compactness argument, $\alpha_T = 1$ implies that there is a linear subspace F_T, $\dim F_T \geqq 7n/8$, such that $\| T \mid F_T \|_2 = 1$. Thus we have

$$1 = \| T \mid F_T \|_2 \leqq C_2 n^{-1/2} \gamma_2(T, X_n^1).$$

The case of an operator T with $\alpha_T \neq 1$ easily follows from the previous one by a homogeneity argument, which concludes the proof of the proposition in the case $p = 1$.

To prove the proposition for $p \in (1, 2)$ define $B_{X_n^p} = Q_n(B_{2n}^p)$ for every $p \in (1, 2)$ and every $n \geqq 2$, where Q_n is the same quotient map as in the first part of the proof, but considered as a map from l_{2n}^p rather than l_{2n}^1 and define $\| \ \|_{X_{2n}^p}$ to be the norm on $\mathbf{R}^n$ with $B_{X_n^p}$ as the unit ball.

(i) is satisfied by the definition of $X_n^p = (\mathbf{R}^n, \| \ \|_{X_n^p})$.

(ii) Since $B_{2n}^1 \subset B_{2n}^p \subset (2n)^{1-1/p} B_{2n}^1$ for $p \in (1, 2)$, by (ii) for $p = 1$, we infer that

$$\begin{aligned} Q_n(B_{2n}^1) &\subset Q_n(B_{2n}^p) \subset (2n)^{1-1/p} Q_n(B_{2n}^1) \\ &= (2n)^{1-1/p} B_{X_n^1} \subset (2n)^{1-1/p} B_n^2 \end{aligned} \tag{3.10}$$

for $p \in (1, 2)$, which in particular means that

$$(2n)^{1/p-1} \| x \|_2 \leqq \| x \|_{X_n^p} \qquad \text{for every } x \in \mathbf{R}^n \text{ and every } p \in (1, 2).$$

The inequality

$$\| x \|_{X_n^p} \leqq \| x \|_p \qquad \text{for every } x \in \mathbf{R}^n \text{ and every } p \in (1, 2)$$

follows from the fact that, by (ii) for $p = 1$, all the vectors from the standard unit vector basis in $\mathbf{R}^n$ are contained in the set $\{Q_n e_j : j = 1, 2, \ldots, 2n\}$. The right hand side inequality in (ii) is trivial. Thus (ii) has been proved.

To prove (iii) note that if E satisfies (iii) for $p = 1$, then for every $x \in E$ we have, by (3.10),

$$\| x \|_{X_n^p} \geqq (2n)^{1/p-1} \| x \|_{X_n^1} \geqq (2C_1)^{-1} n^{1/p-1/2} \| x \|_2 \qquad \text{for } p \in (1, 2),$$

while, by (ii) for $p \in [1, 2)$,

$$\| x \|_{X_n^p} \leqq \| x \|_p \leqq n^{1/p-1/2} \| x \|_2 \qquad \text{for every } x \in \mathbf{R}^n,$$

which completes the proof of (iii) with the constant $2C_1$.

To prove (iv) fix an arbitrary $T \in L(\mathbf{R}^n)$ and let λ_T, V_T and E_T be such that (iv) for $p = 1$ is satisfied with the constant c from Proposition 2.1.

By (3.10), we have

$$\| T \|_{X_n^p} \geqq \| T : X_n^1 \to (\mathbf{R}^n, (2n)^{1-1/p} \| \quad \|_{X_n^1}) \| \geqq \tfrac{1}{2} n^{1/p-1} \| T \|_{X_n^1}.$$

Hence

$$\| V_T \mid E_T \|_2 \leqq c n^{-1/2} \| T \|_{X_n^1} \leqq 2c n^{1/2-1/p} \| T \|_{X_n^p},$$

for every $p \in (1, 2)$. Thus (iv)(a) and (c) are satisfied with the constant $2c$. To prove (iv)(b) assume that

$$| \lambda_T | > (4cC_1 + 1) \| T \|_{X_n^p} \qquad \text{for some } p \in (1, 2).$$

Then, for every $x \in \mathbf{R}^n$, we have

$$\begin{aligned} \| V_T x \|_{X_n^p} &= \| (T - \lambda_T \mathrm{Id}_{\mathbf{R}^n}) x \|_{X_n^p} \\ &> ((4cC_1 + 1) \| T \|_{X_n^p} - \| T \|_{X_n^p}) \| x \|_{X_n^p} \\ &\geqq 4cC_1 \| T \|_{X_n^p} \| x \|_{X_n^p}. \end{aligned}$$

On the other hand, by (ii) and (iii) for $p \in [1, 2)$, we infer that for every $x \in E \cap E_T$ (note that $\dim(E \cap E_T) \geqq 3n/8$)

$$\| V_T x \|_{X_n^p} \leqq n^{1/p-1/2} \| V_T x \|_2 \leqq n^{1/p-1/2} \| V_T \mid E_T \|_2 \| x \|_2$$

$$\leqq 2c \| T \|_{X_n^p} \| x \|_2 \leqq 4cC_1 \| T \|_{X_n^p} n^{1/2-1/p} \| x \|_{X_n^p} \leqq 4cC_1 \| T \|_{X_n^p} \| x \|_{X_n^p},$$

a contradiction which completes the proof of (iv)(b) with the constant $4cC_1 + 1$.

(v) with the constant $2C_2$ follows easily from (v) for $p = 1$ and the inequality

$$\gamma_2(T, X_n^1) \leqq (2n)^{1-1/p}\gamma_2(T, X_n^p) \qquad \text{for every } T \in L(\mathbf{R}^n),$$

which is an easy consequence of (3.10).

To complete the proof of the proposition it is enough to set

$$C = 4\max\{c, C_1, C_2, cC_1 + 1\}.$$

In the sequel we shall need the following "dual" version of Proposition 3.1.

PROPOSITION 3.2. *There is a numerical constant $C \geqq 1$ such that for every $n \geqq 2$ and every $q \in (2, \infty]$ there is a norm $\| \ \|_{Y_n^q}$ on $\mathbf{R}^n$ satisfying the conditions*

(i) *the Banach space $(\mathbf{R}^n, \| \ \|_{Y_n^q})$ is isometrically isomorphic to a subspace of l_{2n}^q,*

(ii) *for every $T \in L(\mathbf{R}^n)$ there is a $\vartheta_T \in \mathbf{R}$, $V_T \in L(\mathbf{R}^n)$ and a linear subspace $E_T \subset \mathbf{R}^n$, $\dim E_T \geqq 7n/8$, such that for every $q \in (2, \infty]$*

(a) $T = \vartheta_T \mathrm{Id}_{\mathbf{R}^n} + V_T$,

(b) $|\vartheta_T| \leqq C \| T \|_{Y_n^q}$,

(c) $\| V_T \mid E_T \|_2 \leqq Cn^{1/q-1/2} \| T \|_{Y_n^q}$.

(iii) *for every $T \in L(\mathbf{R}^n)$ there is a linear subspace $F_T \subset \mathbf{R}^n$, $\dim F_T \geqq 7n/8$, such that for every $q \in (2, \infty]$*

$$\| T \mid E_T \|_2 \leqq Cn^{1/q-1/2}\gamma_2(T, Y_n^q).$$

PROOF. For every $n \geqq 2$ and every $q \in (2, \infty]$ define the norm $\| \ \|_{Y_n^q}$ on $\mathbf{R}^n$ by

$$\| \ \|_{Y_n^q} = \| \ \|_{X_n^p}^* \qquad \text{where } 1/q + 1/p = 1,$$

where X_n^p is the space from Proposition 3.1 and the duality is given by the standard scalar product on $\mathbf{R}^n$. Thus (i) is fulfilled. To prove (ii) fix $T \in L(\mathbf{R}^n)$ and define $\vartheta_T = \lambda_{T^*}$ and $V_T = (V_{T^*})^*$, where λ_{T^*} and V_{T^*} are taken from Proposition 3.1(iv) for the operator T^*. Obviously (ii)(a) and (b) are satisfied while (ii)(c) means that at least $7n/8$ of s-numbers of the operator V_T are not greater than $Cn^{1/q-1/2} \| T \|_{Y_n^q}$ and immediately follows from the fact that the distributions of s-numbers of V_T and V_{T^*} are exactly the same and from Proposition 3.1(iv)(c). Since for every linear operator $T \in L(\mathbf{R}^n)$ we have that

$$\gamma_2(T, Y_n^q) = \gamma_2(T^*, X_n^p) \qquad \text{where } 1/q + 1/p = 1,$$

the same argument concerning s-numbers yields (iii) and completes the proof of Proposition 3.2.

REMARK 3.3. It may be worth mentioning that due to the fact that all spaces X_n^p for a fixed $n \geqq 2$ are generated by the same quotient map Q_n, every space Y_n^q is isometrically isomorphic to the same n-dimensional subspace of $\mathbf{R}^{2n}$, say E_n, equipped with the corresponding norm $\| \ \|_q$. Also, due to the probabilistic nature of the proof of Proposition 3.1, a "typical" n-dimensional subspace E_n of $\mathbf{R}^{2n}$ will work.

4. Construction of the space Y

The basic idea of our construction is identical with the constructions in [15] and [16]. First we define by induction an increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers and a decreasing sequence of reals $(q_k)_{k=1}^{\infty}$ with $q_k > 2$ for $k \in \mathbf{N}$ in the following way: set $n_1 = q_1 = 4$ and assume that $(n_1, \ldots, n_{k-1})$ and $(q_1, \ldots, q_{k-1})$ have been defined. Define q_k to be any number from the interval $(2, q_{k-1})$ satisfying

$$(*) \qquad n_{k-1}^{1/2 - 1/q_k} \leqq 2.$$

Next define n_k to be any positive integer greater than n_{k-1} satisfying

$$(**) \qquad n_{k-1}^{-1/2} n_k^{1/2 - 1/q_k} > k.$$

Finally, we set

$$(4.1) \qquad Y = \left(\bigoplus_{k=1}^{\infty} Y_{n_k}^{q_k} \right)_{l^2}$$

(the direct sum of the spaces $Y_{n_k}^{q_k}$ from Proposition 3.2 in the sense of l^2). Define P_k to be the natural projection onto the k-th factor, for $k \in \mathbf{N}$. We have $P_k(Y) = Y_{n_k}^{q_k}$. Set $X_k = \ker P_k$ for $k \in \mathbf{N}$. We shall need the following easy fact (cf. [15], [16]).

Lemma 4.1. *For every $k \in \mathbf{N}$ and every linear subspace E of X_k with $\dim E \leqq n_k$ we have*

$$d(E, l^2_{\dim E}) \leqq n_{k-1}^{1/2}.$$

Proof. (A copy of the argument in [15] and [16].) Fix $E \subset X_k$, $\dim E \leqq n_k$. Then

$$(4.2) \qquad E \subset \left(\bigoplus_{j \neq k} P_j E \right)_{l^2} \subset X_k = \left(\bigoplus_{j \neq k} Y_{n_j}^{q_j} \right)_{l^2}.$$

Since, by F. John's theorem, for every finite dimensional Banach space F we have $d(F, l^2_{\dim F}) \leqq (\dim F)^{1/2}$ we infer that, for $j < k$, we have

$$(4.3) \qquad d(P_jE, l^2_{\dim P_jE}) \leqq (\dim P_jE)^{1/2} \leqq n_j^{1/2} \leqq n_{k-1}^{1/2}$$

while, by [6] and (*), for $j > k$, we have

$$(4.4) \qquad d(P_jE, l^2_{\dim P_jE}) \leqq (\dim P_jE)^{1/2-1/q_j} \leqq n_k^{1/2-1/q_{k+1}} \leqq 2 \leqq n_{k-1}^{1/2}.$$

The lemma follows easily from (4.2), (4.3) and (4.4).

5. Proof of Theorem 1.1

Let Y be the space defined in the previous section. Obviously Y is a separable and superreflexive space with a finite dimensional decomposition. This completes the proof of Theorem 1.1(i). To prove Theorem 1.1(ii) we shall need a few more definitions. For every $k \in \mathbf{N}$ we define a function

$$\bar{\varphi}_k : L(\mathbf{R}^{n_k}) \to \mathbf{R}$$

by setting for $T \in L(\mathbf{R}^{n_k})$, $\bar{\varphi}_k(T) = \vartheta_T$, where ϑ_T is an arbitrary fixed number satisfying the conditions of Proposition 3.2(ii) for T. Additionally, for the sake of simplicity, we shall assume that

$$(5.1) \qquad \bar{\varphi}_k(\lambda \operatorname{Id}_{\mathbf{R}^{n_k}}) = \lambda \qquad \text{for every } \lambda \in \mathbf{R}.$$

For every $k \in \mathbf{N}$ define

$$(5.2) \qquad \varphi_k(T) = \bar{\varphi}_k(P_kTP_k) \qquad \text{for every } T \in L(Y),$$

and observe that since $\| P_kTP_k \|_{Y^{q_k}_{n_k}} \leqq \| T \|_Y$, by Proposition 3.2(ii) and the definition of φ_k's, we have

$$(5.3) \qquad |\varphi_k(T)| \leqq C \| T \|_Y \qquad \text{for every } T \in L(Y) \text{ and } k \in \mathbf{N},$$

where C is the numerical constant from Proposition 3.2.

Now, for every free ultrafilter $\mathcal{U}$ on $\mathbf{N}$ define

$$(5.4) \qquad \Phi_{\mathcal{U}}(T) = \lim_{\mathcal{U}} \varphi_k(T) \qquad \text{for every } T \in L(Y)$$

and observe that, by (5.3), the limit above exists. In the sequel we shall need the following

LEMMA 5.1. *For every free ultrafilter $\mathcal{U}$ on $\mathbf{N}$ the function $\Phi_{\mathcal{U}}$ defined above is a nontrivial linear multiplicative functional.*

PROOF. We begin with the multiplicavity of $\Phi_{\mathscr{U}}$. To this end fix an arbitrary free ultrafilter $\mathscr{U}$ on N and $T, S \in L(Y)$. Assume to the contrary that

$$\Phi_{\mathscr{U}}(TS) \neq \Phi_{\mathscr{U}}(T)\Phi_{\mathscr{U}}(S).$$

By (5.4), this implies that there is an $\varepsilon > 0$ and $U \in \mathscr{U}$ such that

$$(5.5) \qquad |\varphi_k(T)\varphi_k(S) - \varphi_k(TS)| > \varepsilon \qquad \text{for every } k \in U.$$

Now, fix $k \in U$ such that $k > 10C^2\varepsilon^{-1} \| T \|_Y \| S \|_Y$, consider Y as an l^2 product of $Y^{q_k}_{n_k}$ and X_k and write operators T, S and TS in matrix form with operator entries;

$$(5.6) \qquad T = \begin{bmatrix} A_T & C_T \\ D_T & B_T \end{bmatrix}^{Y^{q_k}_{n_k}}_{X_k}, \qquad S = \begin{bmatrix} A_S & C_S \\ D_S & B_S \end{bmatrix}, \qquad TS = \begin{bmatrix} A_{TS} & C_{TS} \\ D_{TS} & B_{TS} \end{bmatrix}.$$

Since $A_T = P_kTP_k$, by (5.2) and Proposition 3.2(ii), we infer that

$$A_T = \varphi_k(T)\mathrm{Id}_{\mathbf{R}}\, n_k + V_{A_T}.$$

By the same token

$$(5.7) \qquad A_S = \varphi_k(S)\mathrm{Id}_{\mathbf{R}^{n_k}} + V_{A_S} \quad \text{and} \quad A_{TS} = \varphi_k(TS)\mathrm{Id}_{\mathbf{R}^{n_k}} + V_{A_{TS}}.$$

Multiplying the matrices in (5.6) we get $A_TA_S + C_TD_S = A_{TS}$, which together with (5.7) yields

$$(5.8) \qquad \begin{aligned} &(\varphi_k(T)\varphi_k(S) - \varphi_k(TS))\mathrm{Id}_{\mathbf{R}^{n_k}} \\ &\quad = -\varphi_k(T)V_{A_S} - \varphi_k(S)V_{A_T} - V_{A_T}V_{A_S} + V_{A_{TS}} - C_TD_S. \end{aligned}$$

Note that (5.5) implies that

$$(5.9) \qquad \| (\varphi_k(T)\varphi_k(S) - \varphi_k(TS))\mathrm{Id}_{\mathbf{R}^{n_k}} x \|_2 > \varepsilon \| x \|_2 \qquad \text{for every } x \in \mathbf{R}^{n_k}.$$

On the other hand, observe that, by Lemma 4.1,

$$\begin{aligned} \gamma_2(C_TD_S) &\leqq \| C_T : X_k \to Y^{q_k}_{n_k} \| \, \| D_S : Y^{q_k}_{n_k} \to X_k \| \, d(D_S(Y^{q_k}_{n_k}), l^2_{\dim D_S(Y^{q_k}_{n_k})}) \\ &\leqq \| T \|_Y \| S \|_Y n_{k-1}^{1/2}. \end{aligned}$$

Hence, considering C_TD_S as an operator acting on $\mathbf{R}^{n_k}$, by Proposition 3.2(iii), we infer that there is a linear subspace $F \subset \mathbf{R}^{n_k}$ with $\dim F \geqq 7n_k/8$ such that

$$(5.10) \qquad \| C_TD_S \,|\, F \|_2 \leqq Cn_k^{1/q_k - 1/2} n_{k-1}^{1/2} \| T \|_Y \| S \|_Y.$$

Similarly, using now Proposition 3.2(ii), we get that there are linear subspaces E_{A_T}, E_{A_S} and $E_{A_{TS}}$ all of dimension at least $7n_k/8$ such that

$$
\begin{aligned}
&\| A_T \mid E_{A_T} \|_2 \leqq C n_k^{1/q_k - 1/2} \| T \|_Y, \\
(5.11) \qquad &\| A_S \mid E_{A_S} \|_2 \leqq C n_k^{1/q_k - 1/2} \| S \|_Y, \\
&\| A_{TS} \mid E_{A_{TS}} \|_2 \leqq C n_k^{1/q_k - 1/2} \| T \|_Y \| S \|_Y.
\end{aligned}
$$

Finally, observe that $\dim E \geqq 3n_k/8$, where

$$E = F \cap E_{A_S} \cap E_{A_T} \cap E_{A_{TS}} \cap V_{A_S}^{-1}(E_{A_T}),$$

and note that, by the choice of k, using (5.3), (5.10), (5.11) and (**), for every $x \in E$ we obtain

$$
\begin{aligned}
&\| (- \varphi_k(T) V_{A_S} - \varphi_k(S) V_{A_T} - V_{A_T} V_{A_S} + V_{A_{TS}} - C_T D_S) x \|_2 \\
&\leqq |\varphi_k(T)| \, \| V_{A_S} x \|_2 + |\varphi_k(S)| \, \| V_{A_T} x \|_2 + \| V_{A_T} V_{A_S} x \| \\
&\quad + \| V_{A_{TS}} x \|_2 + \| C_T D_S x \|_2 \\
&\leqq (4C^2 \| T \|_Y \| S \|_Y n_k^{1/q_k - 1/2} + C \| T \|_Y \| S \|_Y n_k^{1/q_k - 1/2} n_{k-1}^{1/2}) \| x \|_2 \\
&\leqq 5C^2 \| T \|_Y \| S \|_Y n^{1/q_k - 1/2} n_{k-1}^{1/2} \| x \|_2 \\
&\leqq \frac{k\varepsilon}{2} \frac{1}{k} \| x \|_2 \\
&= \frac{\varepsilon}{2} \| x \|_2
\end{aligned}
$$

which together with (5.9) contradicts (5.8) and completes the proof of the multiplicativity of $\Phi_{\mathscr{U}}$. The proof of the additivity of $\Phi_{\mathscr{U}}$ goes along the same lines and is much simpler, so we omit it. It remains to see that $\Phi_{\mathscr{U}}$ is a nontrivial linear multiplicative functional but it follows immediately from (5.1) and (5.4) that $\Phi_{\mathscr{U}}(\mathrm{Id}_Y) = 1$, which concludes the proof of the lemma.

Proof of Theorem 1.1(ii). Let $N_1, N_2, \ldots$ be a sequence of infinite disjoint subsets of $\mathbf{N}$ and let, for each $i \in \mathbf{N}$, $\mathscr{U}_i$ be a free ultrafilter on $\mathbf{N}$ such that $N_i \in \mathscr{U}_i$. Define a map $h : L(Y) \to l_{\mathbf{N}}^\infty$ by the formula

$$h(T) = (\Phi_{\mathscr{U}_1}(T), \Phi_{\mathscr{U}_2}(T), \ldots) \qquad \text{for every } T \in L(Y).$$

Since every nontrivial linear multiplicative functional is of norm 1 we infer that $\| h(T) \|_{l_{\mathbf{N}}^\infty} \leqq \| T \|_Y$ for every $T \in L(Y)$. On the other hand, it follows from the lemma above that h is a homomorphism. Moreover, h maps $L(Y)$ onto $l_{\mathbf{N}}^\infty$. Indeed, let $(\alpha_i)_{i=1}^\infty \in l_{\mathbf{N}}^\infty$. For every $i \in \mathbf{N}$ define

$$P_{N_i}((y_k)_{k=1}^{\infty}) = (\mathrm{Ind}_{N_i}(k) y_k)_{k=1}^{\infty} \qquad \text{for every } (y_k)_{k=1}^{\infty} \in Y,$$

where Ind_A denotes the characteristic function of the set A (i.e. P_{N_i} is the natural projection in Y onto a subspace spanned by the coordinates from N_i).

Obviously $\Sigma_{i=1}^{\infty} \alpha_i P_{N_i} \in L(Y)$ and by (5.1) and (5.4) we have

$$h\left(\sum_{i=1}^{\infty} \alpha_i P_{N_i}\right) = (\alpha_i)_{i=1}^{\infty}.$$

The proof of Theorem 1.1(ii) is completed by the remark that the Banach algebras $l_{\mathbf{N}}^{\infty}$ and $C(\beta\mathbf{N})$ are isometrically equivalent.

Proof of Theorem 1.1(iii). Let $\{N_t : t \in \mathbf{R}\}$ be a family of infinite "almost disjoint" subsets of $\mathbf{N}$. For each $t \in \mathbf{R}$ let $\mathcal{U}_t$ be a free ultrafilter on $\mathbf{N}$ with $N_t \in U_t$. Define P_{N_t}, for $t \in \mathbf{R}$, as above. It easily follows from (5.1), (5.4) and the basic properties of free ultrafilters that systems $\{P_{N_t} : t \in \mathbf{R}\}$ and $\{\Phi_{\mathcal{U}_t} : t \in \mathbf{R}\}$ satisfy the requirements of Theorem 1.1(iii), which completes the proof of Theorem 1.1.

6. Remarks

Remark 6.1. The same argument as above yields that the space $Y \times l^2$ satisfies the requirements of Theorem 1.1. (In fact $Y \simeq Y \times l^2$.)

Remark 6.2. Since the space Y constructed here is "essentially" the same as the space in [16] one can prove that Y has no basis.

It is easy to see that if X is a complex Banach space then $L(X)$ does not admit a real linear multiplicative functional. Thus we have

Corollary 6.3. (The real case only). The space Y does not admit a complex structure (cf. [15]).

Remark 6.4. It is well known that there does not exist a linear multiplicative functional on $L(\mathbf{R}^n)$ for every $n \geqq 2$. It may be worth mentioning that this implies that the functions $\bar{\varphi}_k$ in §5 are not linear multiplicative functionals despite the fact that their limit is. Observe that if X is a Banach space and Z is the Cartesian product of n copies of X with $n \geqq 2$, then there is a natural homomorphic embedding h of $L(\mathbf{R}^n)$ into $L(Z)$ with $h(\mathrm{Id}_{\mathbf{R}^n}) = \mathrm{Id}_Z$. Thus we get

COROLLARY 6.5. (Solution of Problems 7.4 and 7.6 in [15]). *The space Y is not isomorphic to any finite Cartesian power of any Banach space.*

Moreover, reasoning along the same lines one can prove

COROLLARY 6.6. (A partial answer to Problem 7.5 in [15]).

(i) *For every $k \in \mathbf{N}$ there does not exist a continuous representation h of the group $O(2k)$ of all linear isometries of l^2_{2k} in $L(Y)$ satisfying the conditions*:

(a) $h(\mathrm{Id}_{l^2_{2k}}) = \mathrm{Id}_Y$,

(b) $h(T_1) + h(T_2) = 0$ *for every* $T_1, T_2 \in O(2k)$ *such that* $T_1 + T_2 = 0$.

(*Note that the condition* (b) *is equivalent to the condition* $h(-\mathrm{Id}_{l^2_{2k}}) = -\mathrm{Id}_Y$.)

(ii) *For every $k \in \mathbf{N}$ there does not exist a continuous representation h of the group $O(2k+1)$ satisfying conditions* (a) *and* (b) *above and*

(c) $h(T_1) + h(T_2) + h(T_3) = \mathrm{Id}_Y$ *for every* $T_1, T_2, T_3 \in O(2k+1)$ *such that* $T_1 + T_2 + T_3 = \mathrm{Id}_{l^2_{2k+1}}$.

Since the cardinality of the set of all linear multiplicative functionals on $C(\beta\mathbf{N})$ is equal to the cardinality of $\beta\mathbf{N}$ $(=2^{\mathfrak{c}})$ we get

COROLLARY 6.7. *The cardinality of the set of all linear multiplicative functionals on $L(Y)$ is equal to the cardinality of the set of all continuous linear functionals on $L(Y)$* $(=2^{\mathfrak{c}})$.

REMARK 6.8. Proposition 3.2 can be viewed as a tiny step towards understanding the following well known problem: Does there exist an infinite dimensional Banach space X with the property that every $T \in L(X)$ is of the form $\lambda\,\mathrm{Id}_X + K$ with K being a compact (resp. nuclear) operator? In [13] a construction of a nonseparable space is given on which every continuous linear operator is a "separable perturbation" of a multiple of the identity operator.

REMARK 6.9. For a Banach space X denote by $\mathcal{M}(X)$ the minimal closed ideal in $L(X)$ containing all operators of the form $TS - ST$ where $T, S \in L(X)$. Clearly $L(X)/\mathcal{M}(X)$ is a commutative Banach algebra and $\mathcal{M}(X) = L(X)$ iff $L(X)$ does not admit a linear multiplicative functional. Therefore, Problems A and B can be reformulated to find a (reflexive) Banach space X with "relatively large" quotient algebra $L(X)/\mathcal{M}(X)$. Thus, by Theorem 1.1(ii) we have

COROLLARY 6.10. *The Banach algebra $L(Y)/\mathcal{M}(Y)$ admits a homomorphism onto $C(\beta\mathbf{N})$.*

REMARK 6.11. In discussions with the author, P. Wojtaszczyk observed that one can construct a nonreflexive separable Banach space satisfying conditions (i)–(iii) of Theorem 1.1 using the spaces considered in [11].

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